Modeling Banjo Strings as Weakly Coupled and Weakly Damped Oscillators

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We mathematically model the non-normal decay of a plucked banjo strings. The model assumes that the non-normal behavior in the decay of a vibrating banjo string is due to the coupling between the horizontal and vertical motion of the string. We then compared experimental and theoretical plots for the decay, and ultimately found that our model, which is based on weakly coupled and weakly damped coupled oscillators, could not accurately describe the sound produced by a banjo string, and could only be used to approximate local sections in the global decay curve.

I. INTRODUCTION

With the advent of digitization in the music industry, there have been attempts to reproduce analog sounds in the digital realm. It is often possible to create digital models that mimic their analog counter parts, but creating emulations becomes difficult if the analog equipment exhibits unusual, non-linear or non-normal behavior. In order to accurately recreate an analog sound, there needs to be an understanding of timbre, overtones, amplitude decay, etc. Some instruments, such as the American 5 string banjo, exhibit unusual sound decay, which contributes to the difficulty in digitizing the banjo. There is a small but growing, body of research on the physics of instruments similar to the banjo, which may lead to insights that could benefit the music industry as well as physicists interested in coupled oscillators.

II. BACKGROUND

The American banjo is an instrument that varies by size and number of strings. The standard banjo has metal strings that are coupled to a thin membrane called the drum head through a wooden bridge, as shown in figure 1. It is known that banjo strings exhibit unusual behavior. The decay of the sound produced by the strings is often observed as "non-normal," which is short hand to describe periodic, non-exponential behavior. This behavior is shown in figure 3 (Stephey et. al), which exhibits how the sound of a single pluck changes over time. As noted in Stephey's paper, banjo strings generally exhibit three kinds of decay for the harmonics, two of which can be described with the usual decaying exponential.

$$P = P_0 e^{-\alpha t} \tag{1}$$

where P is power, P_0 is initial power, α is the decay constant and t is time. We focus on the third kind of behavior, which is the cyclical, non exponential behavior.

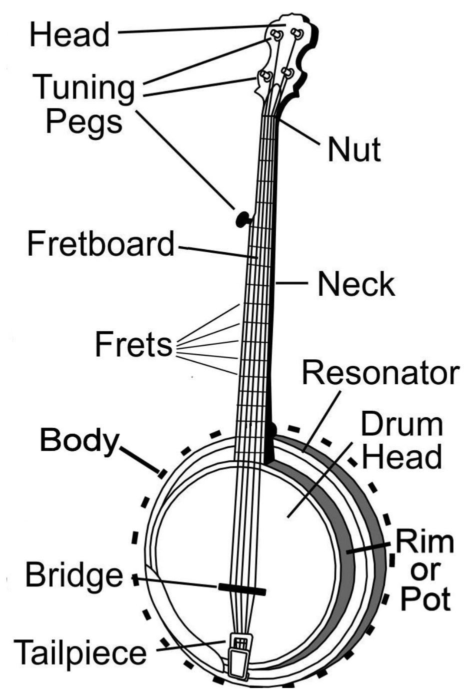


FIG. (1) Schematic of a banjo. The drum head is a thin membrane.

TABLE I. The standard G-tuning of a five string banjo.

String	Note	Fundamental frequency (±0.1 Hz)
1	D	293.7
2	B	246.9
3	G	196.0
4	D	146.8
5	G	392.0

FIG. (2) Taken from Stephey et. al. Information about the banjo strings.

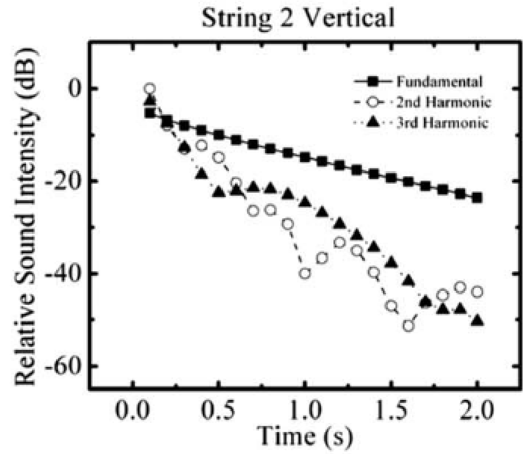


FIG. (3) Power time series for the three harmonics produced by the pluck of a single banjo string. String 2 is defined in table 2. Second harmonic of the pluck exhibits the characteristic non-normal behavior expected in a banjo. Taken from Stephey et. al. More information on the other kinds of decay can be found in the paper.

At best, the literature on this topic provide suggestions for the sources of this non-normal behavior. We first define the motion of a plucked banjo string to be the linear superposition of its horizontal and vertical motion, where vertical motion is perpendicular to the drum head. One explanation for the unusual behavior is that there may be a slight difference in phase between the sound generated by the vertical and horizontal motions of string vibration, leading to an interference pattern that is cyclical and seemingly non-exponential. The other explanation proposes that the vertical and horizontal motions of the strings are coupled. When the string vibrates, it produces a cyclical transfer of energy to between the different deconstructed motions of the strings. We use our experiment to explore the latter proposition with the intent of providing insight on the physics behind non-normal decay in vibrating banjo strings, hoping to merge and expand the current literature.

III. THEORY AND MODELING

Like any physical structure, we expect to be able to describe a banjo string by its set of normal modes. Recall that a normal mode is a special case of vibration that describes collective motion of an object, which is a string in our case. For physical strings, we expect each mode to be a degenerate pair that corresponds to the horizontal and vertical motion of the string.

In a simple approximation, we might expect the polarized motions to be completely independent. However, Politzer argues that by casual inspection it is clear that the horizontal and vertical motions of the string are weakly coupled. Politzer's argument is supported by work done by Stephey et al in an experiment dealing with banjo strings. Whenever they attempted to isolate the motion of a string (i.e. pluck in a single direction), they would consistently find slight elliptical motion regardless of initial pluck orientation, which suggests that vertical and horizontal motions of a string are weakly coupled. In addition we expect weak damping to some extent, since musical instruments generally have sustained notes. We therefore have a general idea that our system can be described as weakly coupled and weakly damped. We can create a mathematical model that assumes coupling is the source of the banjo's unusual behavior. However, developing a thorough analytical model requires rigorous steps that are either impossible to carry out or very difficult (Politzer). Politzer develops a model for a simple system that could be used to describe a string that is fixed at two points, and we adopt this model to describe this experiment's physical system. We define the motion of the string as a vector whose components $x_1(t)$ and $x_2(t)$ respectively describe the vertical and horizontal motion of a string.

$$\mathbf{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{2}$$

We find that the equation of motion for a single string is the following:

$$\ddot{\mathbf{x}} = -\begin{bmatrix} 1 & \epsilon \\ \epsilon & 1 \end{bmatrix} \mathbf{x} - \begin{bmatrix} \gamma & 0 \\ 0 & 0 \end{bmatrix} \dot{\mathbf{x}} \ . \tag{3}$$

where x is a vector whose components are the horizontal and vertical motion of the string, ϵ is the coupling constant, and γ is the damping constant for motion in the vertical direction. We do not take into account damping in the horizontal direction to stay consistent with Politzer, who argues the damping in the horizontal direction is negligible. Issues with this assumption are discussed in sections below. We accept this assumption because string damping is more effective for the direction that is perpendicular to the plane of the drum head, since the drum head was built to vibrate in that direction. Equation 3 details the motion of a string using a linear matrix transformation. The matrices represent zeroth order

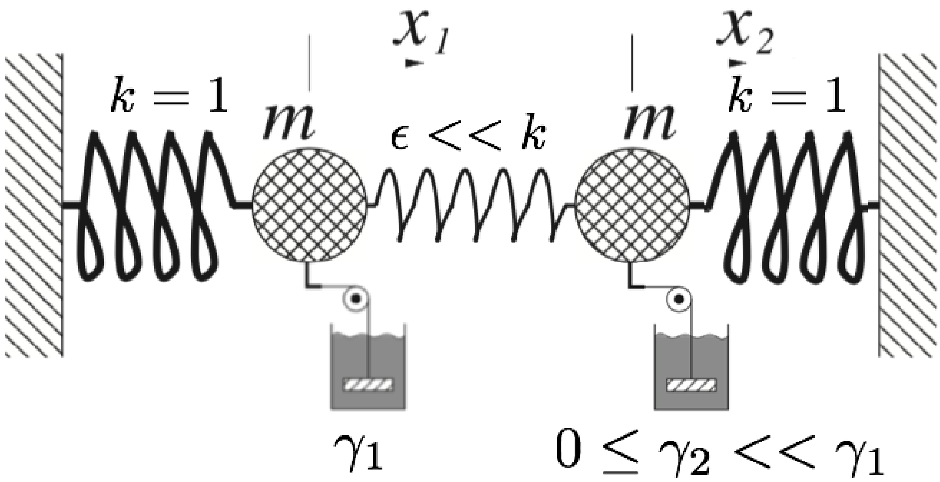


FIG. (4) Two weakly coupled and weakly damped masses on springs, which serve as an analogy to a system where the vertical and horizontal motions of a string are coupled and damped. In this analogy, k is the spring constant, x_1 is horizontal displacement of the mass on the left, x_2 is the horizontal displacement of the mass on the right, γ_1 and γ_2 are the damping constants. In the experiment, we assume γ_2 is zero. In this analogy the motion is dampened by the drag in a viscous fluid. Notice that ϵ, the coupling constant, is much less than the spring constant. The motions of the two masses represent the horizontal and vertical motion of a string in a banjo.

restoring forces for coupling and damping that affect the horizontal and vertical motions of a string. This model is analytically equivalent to figure 4, which provides conceptual insight as to how we are modeling our system. In our model we assume that the damping in horizontal direction is negligible, which we explain further in below sections. Therefore, we are modeling our physical system as two weakly damped and weakly coupled oscillators, where each oscillator corresponds to the vertical and horizontal motions of a banjo string.

We then find that power dissipated is the following:

$$P_{\text{inst}} = \gamma \dot{x_1}(t)^2 \tag{4}$$

where P_{inst} is instantaneous value of lost power in the mathematical two oscillator model. Recall that we assume that energy is only being dissipated in the vertical direction, which is why we only see a dependence on $\dot{x_1}$ in equation 4. With this mathematical model, we

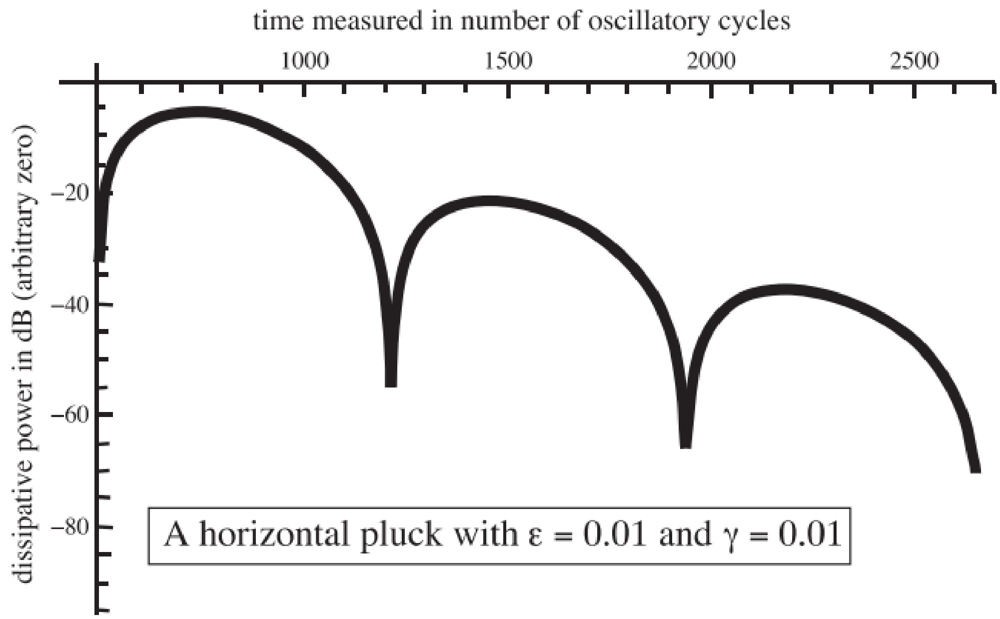


FIG. (5) This is a log plot of the power dissipated of the mathematical system derived from 3. The system is weakly coupled and weakly damped as shown by the values of the coupling constant ϵ and the damping constant γ. We display the log of the power dissipated because our experimental devices measure the log of data we obtain.

expect to observe a cyclical pattern of growth and decay that is "non-normal", as shown in figure 5. The figure displays the non-normal behavior for a string that is initially plucked in the horizontal direction. This means that over time, energy should be transfered to the vertical motion of the string due to coupling. This provides an interesting conceptual insight because we observe an initial growth in power dissipated, which corresponds to the increase in motion in the vertical direction due to coupling. The growth is followed by natural decay of the string due to damping. The pattern of growth and decay repeats, and produces the bizarre behavior of cyclical non-exponential decay, which resembles what has been observed in current literature.

IV. EXPERIMENTAL METHODS

The experiment outlined below is based on the idea that if our mathematical model for coupled motion accurately describes the reality of a banjo string's motion, then that further suggests that coupling is indeed the source of the non-normal behavior.

A. Experimental Set Up

In this experiment we record and analyze the sound of a vibrating banjo string. We used a Shure SM57 microphone, a Scarlett 2i2 audio interface, a microphone stand, a computer, and an American 5 string banjo. We laid the banjo flat on a table on top of a soft case. We placed it on the soft case because the banjo, when plucked, would create extraneous noise due to its vibration on a hard surface. We placed the microphone approximately 7cm away from the bridge within the plane of the drum head. We were able to adjust the vertical distance from the drum head by adjusting the microphone stand. Our set up is summarized in figure 6. We connected the microphone to the an audio interface using an XLR cable, which we then connected to a computer via USB. We used Labview to record the signal from the microphone, and were then able to analyze the data in Mathematica. With our set up, it was easy to compare the decay of different harmonics for a single pluck. In addition, using the microphone stand, we were able to easily vary the microphone height from the drum head and observe how height impacted the appearance of decay in a banjo string.

Our general procedure began by analyzing how height affects the signal. We wanted to find the microphone location where non-normal behavior was most clearly observable while simultaneously test the hypothesis that height should affect the appearance of non-normal behavior.

Once we determined the optimal height for observing non-normal behavior, we fixed the microphone stand at the desired height, and recorded data for each of the strings at that height. We only plucked one string at a time and muted all other strings. In order to be consistent with our plucking, we used human hair to pluck the strings vertically. We assumed that the breaking point of hair is sufficiently constant for our purposes. We would loop the hair around the target banjo string, and pull until the hair broke. The pluck would be recorded for about 15 seconds to obtain as much data of the decay as we could.

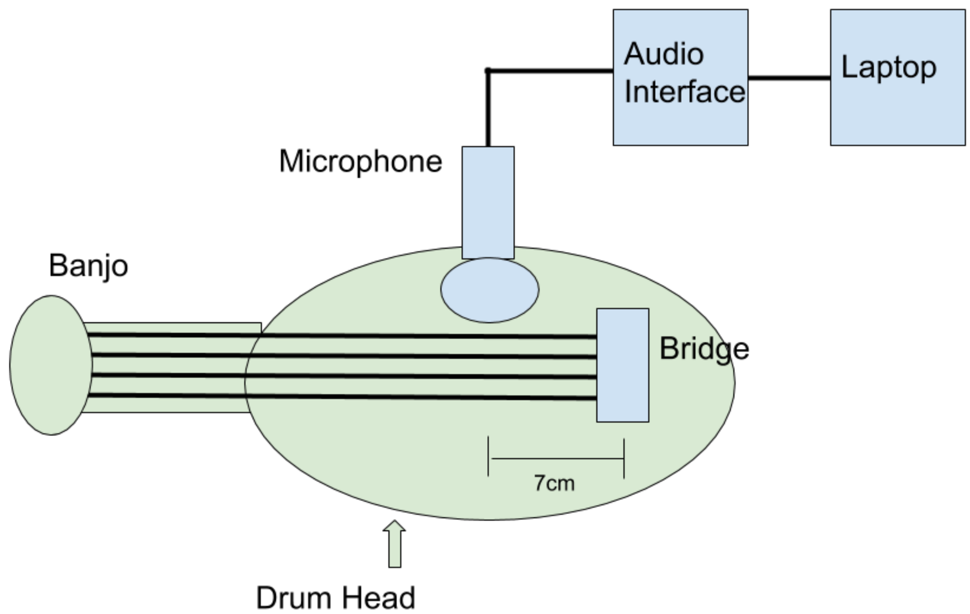
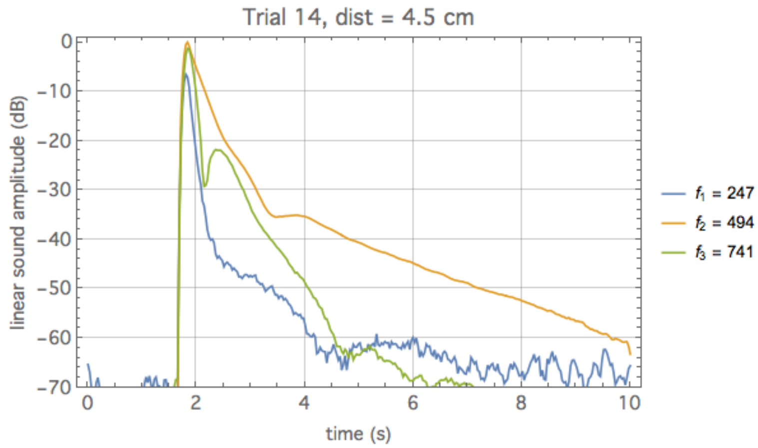


FIG. (6) This is a conceptual summary of the experimental set up. We do not display the microphone stand. In reality we can adjust the vertical distance of the microphone from the drum head.

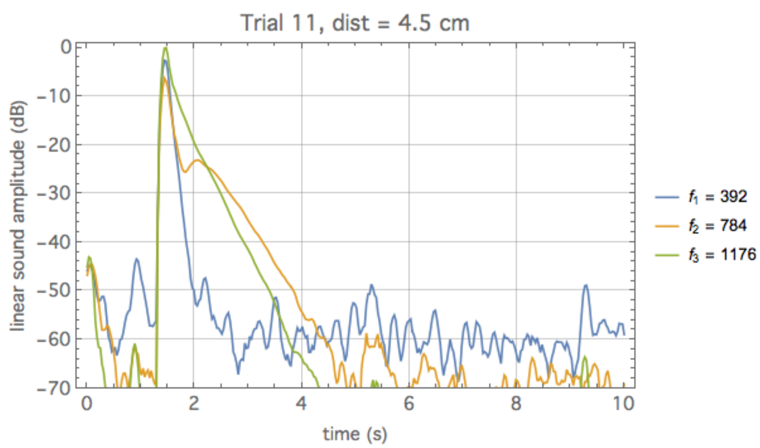
V. RESULTS

We found that varying the microphone height does indeed change the appearance of the decay, as shown in figure 8, where the change is most conspicuous with the third harmonic. It appears that as we increased the microphone height, the observed non-normal behavior occurred at later points in the overall decay. We found that non-normal behavior is most clearly visible at 4.5 centimeters, which is where the microphone was closest to the banjo. We used this height for consequent harmonic analysis of the other strings, described in the following.

We found that our mathematical model could be used to describe small intervals of time in our experimental data, but ultimately failed to describe the complete experimental time series. With our current model, the only way we can accurately describe the reality of a banjo's decay is if the parameters in our model evolved over time, which is not an idea we have considered or will consider in our experiment. These conclusions can be made from comparing the mathematical model, shown in 9, against select time intervals in the experimental plots, shown in 7 and 8. We specific comparisons in section VI.

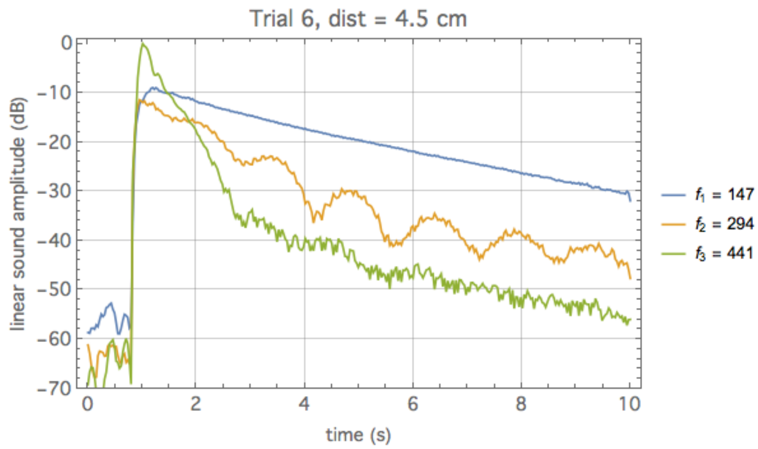


(a) Plot of decay for string with frequency 247.

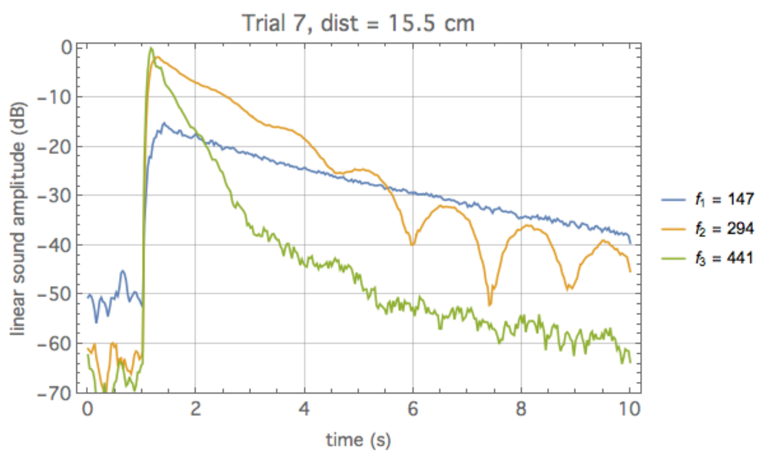


(b) Plot of decay for string with frequency 392.

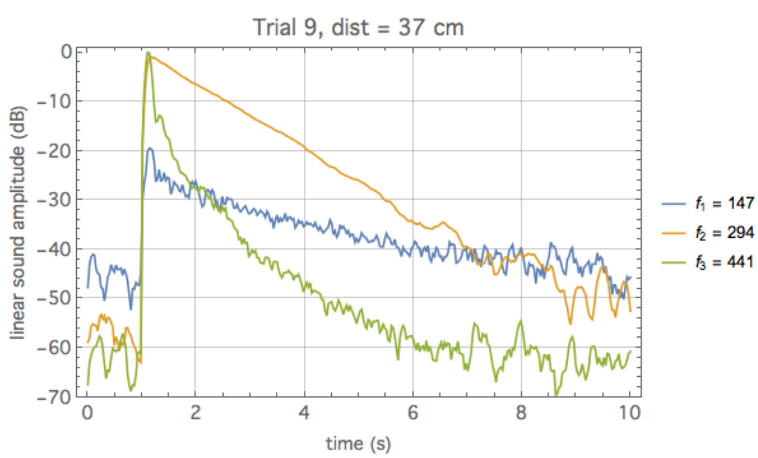
FIG. (7) A plot when microphone is located 4.5 cm above drum head for different banjo strings. It depicts decay behavior for the first three harmonics for a particular banjo string.



(a) A plot when microphone is located 4.5 cm above drum head.

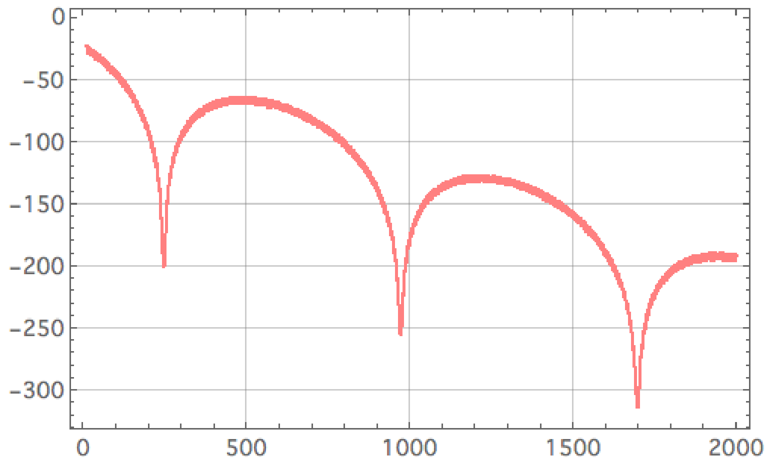


(b) Plot when microphone is located 15.5 cm above drum head.

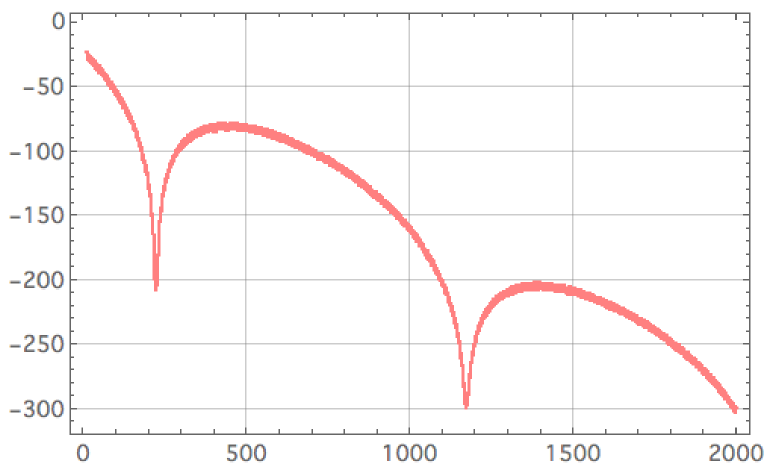


(c) Plot when microphone is located 37 cm above drum head.

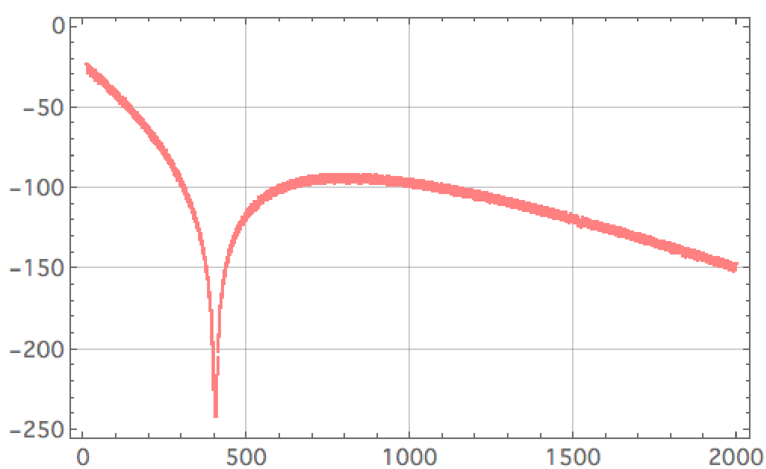
FIG. (8) A plot when the same banjo string is plucked at variable microphone heights. Plots depicting decay behavior for the first three harmonics of a particular pluck of a banjo string. We clearly observe non-normal behavior in the second harmonic for all the graphs.



(a) A plot of mathematical model when $\epsilon = .01$, $\gamma = .01$



(b) A plot of mathematical model when $\epsilon = .01$, $\gamma = .015$



(c) A plot of mathematical model when $\epsilon = .005$, $\gamma = .01$

FIG. (9) Plots produced from our mathematical at different values for the coupling and damping constant. These values create plots that resemble features in the plots produced from our experimental data.

VI. DISCUSSION AND CONCLUSION

For particular values for the damping and coupling constant, our mathematical model, shown in figure 9a, resembles the experimental decay of the third harmonic in figure 8a. Similar observations can be made when comparing the model shown in 9c against the third and fundamental harmonic in figure 7a and the third harmonic in figure 7b. Additionally, if we examine the time interval of 6 seconds to 10 seconds of figure 8b, we find that there are parameter values that produce a decent approximation of reality, shown in 9b. However, in most cases, our mathematical model does not describe the complete evolution of string decay.

Although the mathematical model is able to approximate the cyclical non-exponential behavior for selected time intervals in our experimental data, it does not accurately describe the full picture. More rigorous analysis is needed before determining whether coupling is the main contributor to the observed non-normal behavior. Additionally, we did observe that the appearance of the decay changes alongside the microphone height. Our current mathematical model can not account for these changes in appearance. It may be the case that our model was not rigorous enough and may be built on loose assumptions. One of our initial assumptions was that we could approximate the system with the a simple linear matrix transformation, which Politer notes is a very simplified model. Our model is built on a zeroth order approximations of the damping and coupling restoring forces, which is likely not appropriate to describe reality. In addition, our model assumed that all the energy dissipated is from the vertical motion of the string, which was justified by arguing that dissipation in horizontal direction is negligible. Non-negligible dissipation in horizontal direction could change the appearance of decay, since it would change the interplay of energy transfer between the horizontal and vertical motions of the string. It may also be the case that coupling simply isn't the entire picture. This warrants further exploration of the argument in Stephey et al that the phase difference between the horizontal and vertical motion of the strings are the cause behind the non-normal behavior. Our experiment verifies the non-normal behavior in banjo strings, and provides insight suggesting the need for a more rigorous model and the need to explore alternative contributors to the banjo's unusual behavior.

VII. ACKNOWLEDGEMENTS

I would like to thank my team members Sasha Greenfield, Sophie Rooks, and my advisor Melissa Eblen-Zayas for all their contributions.

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